Note on Topology

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1 Point-Set Topology

2 Convex set, combination, hull and cone

Definition 2.1 (Convex set)

A set $S \subset R^n$ is convex if $\forall \lambda \in [0,1]$ and $\forall x, x' \in S$, $\lambda x + (1-\lambda)x' \in S$.

Lemma 2.1 (Convex set's property)

1. *$C :=$ convex set, $\beta :=$ real number, then $\beta C = \{x : x = \beta c, c \in C\}$ is convex.*
2. *$C, D :=$ convex set, then $C + D = \{x : x = c + d, c \in C, d \in D\}$ is convex.*
3. *$S, T :=$ convex set, then $S \cap T$ is a convex set.*

Lemma 2.2

$C :=$ convex set, $y :=$ a point exterior to the closure of C. Then there is a vector a such that $a^T y < \inf_{x \in C} a^T x$.

Proof

$$a^T y < \inf_{x \in C} a^T x \iff \inf_{x \in C} (a^T x - a^T y) > 0 \iff \inf_{x \in C} a^T (x - y) > 0$$

That is equal to show that there exists $a^T(x-y) > 0$, i.e., the included angle is acute. Define $f(x) = ||x - y||$ (norm/distance). We want to $\min_{x \in C} f(x)$, find the point $x \in C$ closer to y. Since C is closure, there must be an optimal solution x^0, and $||x^0 - y|| \leq ||x - y|| \quad \forall x \in C$. Given x_0, let $x \in C$, then $\forall 0 < \alpha < 1$, $x_0 + \alpha(x - x_0) \in C$ (Convex set definition). And

$$||x_0 + \alpha(x - x_0) - y|| \geq ||x_0 - y||$$

Expanding the inequation then we have $\alpha||x - x_0||^2 + 2|x_0 - y|^T(x - x_0) \geq 0$, let $\alpha \to 0$, we have $|x_0 - y|^T(x - x_0) \geq 0$, that is

$$(x_0 - y)^\top x \geqslant (x_0 - y)^\top x_0 = (x_0 - y)^T (x_0 - y + y) = (x_0 - y)^\top (x_0 - y) + (x_0 - y)^\top y$$

Let $a = (x_0 - y)$, we have $a^\top x \geqslant a^T a + a^\top y$, since $a^T a$ is positive, a is what we want.

■

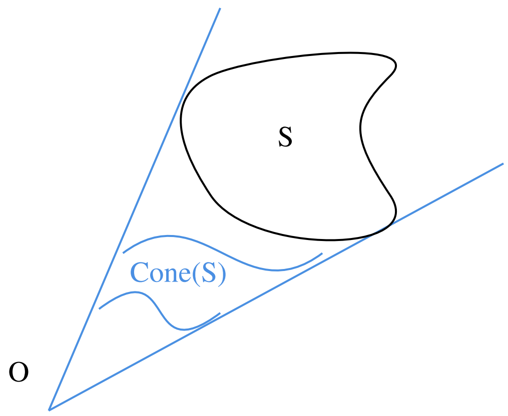

Figure 1: Conic Hull

Definition 2.2 (Convex combination)

$y = \sum_{i=1}^{m} \lambda_i y_i$ *is a convex combination of* $y_1, ..., y_m$ *if* $\lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1$.

Definition 2.3 (Cone, Convex Cone)

1. $C \subset R^n$ *is a cone if* $\forall x \in C, \alpha > 0$, $\alpha x \in C$.
2. $C \subseteq R^n$ *is a convex cone if* $\forall x, y \in C, \alpha, \beta \geq 0$, $\alpha x + \beta y \in C$.

Example 2.1

Definition 2.4 (Convex hull)

Q *is a convex hull of* $v_1, .., v_k$ *if* $Q = \{v \in \Re^n : v$ *is a convex combination of* $v_1, v_2, \ldots, v_k\}$, *and we write* $Q = \operatorname{conv}(v_1, v_2, \ldots, v_k)$.

Note on *The convex hull of* $S \subseteq \mathbb{R}^n$ *is the smallest convex set containing* S.

Property

1. *Intersection of all convex sets containing* S.
2. *The set of all convex combinations of points in* S.

Theorem 2.1 (Convex set and convex hull)

A set is convex iff convexhull(S)=S.

Definition 2.5 (Conic Hull, Closure of Cone)

1. *Given a set* S, *the conic hull of* S, *denoted by* $\text{cone}(S)$, *is the set of all conic combinations of the points in* S, *i.e., the smallest convex cone included* S.

$$\text{cone}(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \geq 0, x_i \in S \right\}$$

2. *Closure of* $\text{cone}(S) :=$ *the closed convex hull of* S.

Note on *Conic hull is convex and includes the zero point.*

Lemma 2.3

A closed bounded convex set in R^n is equal to the closed convex hull of its extreme points.

3 Hyperplane and Polytope

Theorem 3.1 (Projection Lemma?)

Let $X \in R^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y, moreover, for all $x \in X$ we have $(y - x^*)^{\mathrm{T}} (x - x^*) \leq 0$.

Definition 3.1 (Hyperplane)

1. *A set $H \subset R^n$ is a hyperplane := $H = \{x \in R^n : \sum_{i=1}^n \alpha_i x_i = \beta\}$ for some $\beta \in R$ and some $(\alpha_1, \ldots, \alpha_n) \subset R^n$ such that $\alpha_i \neq 0$ for some i.*
2. *Positive half space of H: $H^+ := \{x \in R^n : \sum_{i=1}^n \alpha_i x_i \geqslant \beta\}$.*
3. *Negative half space of H: $H^- := \{x \in R^n : \quad \sum_{i=1}^n \alpha_i x_i \leqslant \beta\}$*

Example 3.1

1. If $n = 1$, H contains the point $\frac{\beta}{\alpha}$.
2. If $n = 2$ and $\alpha_1, \alpha_2 \neq 0$, H is the line $\alpha_1 x_1 + \alpha_2 x_2 = \beta$.

Property

1. *$H^+ \cup H^- = R^n, \quad H^+ \cap H^- = H$*
2. *A hyperplane H and its associated half spaces H^+ and H^- are convex sets.*

Lemma 3.1

C := convex set, y := a boundary point of C. Then there is a hyperplane containing y and containing C in one of its closed half space.

Proof Let H starts as the sequence of $\{y_0, y_1, \ldots, y\}$, according to lemma 2.2, $\forall y_k$, we have $a_k^T y_k < \inf_{x \in C} a_k^T x$, and converge to y we have $a^T y < \inf_{x \in C} a^T x$, that is, $\forall y_k$, $a_k = x_{0k} - y_k$, and converge to for y, $a = 0$. The hyperplane $a^T y$ is what we want. ∎

Theorem 3.2 (Separating Hyperplane Theorem (ali_ahmadi_orf_2016))

1. *If S and T are two disjoint convex sets in R^n then there is a hyperplane $H \subset R^n$ such that $S \subset H^+$ and $T \subset H^-$.*
2. *Let C and D be two convex sets in R^n that do not intersect (i.e., $C \cap D = \emptyset$). Then, there exists $a \in R^n$, $a \neq 0$, $b \in R$, such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.*
3. *Special Case: Let C and D be two closed convex sets in R^n with at least one of them bounded, and assume $C \cap D = \emptyset$. Then $\exists a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R}$ such that*

$$a^T x > b, \forall x \in D \text{ and } a^T x < b, \forall x \in C$$

Note on Equality *Note that the equality in this theorem cannot be neglected (**ali_ahmadi_orf_2016**).*

For example, for $A = (x, y) : y \geq 0 \forall x \leq 0, y > 0 \forall x > 0$, then we can find $a = (1, 0)^T, b = 0$ to separate $A, \overline{A}$. However, there does not exists such a, b to separate withtout equality. The case of strict separate, i.e., $a^T x < b$ and $a^T x > b$ hold simultaneously, may not exist.

Proof [Special Case (**ali_ahmadi_orf_2016**)] Let $c \in C$ and $d \in D$ be the points with the minimal distance, i.e.,

$$\begin{aligned} \operatorname{dist}(C, D) = & \inf \|u - v\| \\ & \text{s.t. } u \in C, v \in D \end{aligned}.$$

Furthermore, let

$$a = d - c, b = \frac{\|d\|^2 - \|c\|^2}{2}.$$

Then $f(x) = a^T x - b$ is the separating hyperplane what we want. We claims that

$$f(x) > 0, \forall x \in D \text{ and } f(x) < 0, \forall x \in C.$$

Note that we choose a to be perpendicular to dc, and b to ensure

$$f\left(\frac{c+d}{2}\right) = (d - c)^T \left(\frac{c+d}{2}\right) - \frac{\|d\|^2 - \|c\|^2}{2} = 0.$$

Then we can prove $f(x) > 0, \forall x \in D$ and $f(x) < 0, \forall x \in C$. Suppose for the sake of contradiction that $\exists \bar{d} \in D$ with $f(\bar{d}) \leq 0$, i.e., $(d - c)^T \bar{d} - \frac{\|d\|^2 - \|c\|^2}{2} \leq 0$. Since for $g(x) = \|x - c\|^2$, $\nabla g^T(d)(\bar{d} - d) < 0$, we can find shorter distance, and this contradicts our assumption. ■

Corollary 3.1 (Separate point and convex set)

Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $x \in R^n$ a point not in C. Then x and C can be strictly separated by a hyperplane.

Note on Special case: convex cone *Particularly, if C is a convex cone, then we can find a horizontal plane through the origin to separate C and any point outside C, i.e., for any $x \notin C$, there exists nonzero $d \in R^n$ such that $d^T x < 0$ ($d^T y \geq 0$) for all $y \in C$.*

Definition 3.2 (Supporting hyperplane)

A hyperplane containing a convex set C in one of its closed half spaces, and containing a boundary point of C.

Lemma 3.2

Let C be a convex set, H a supporting hyperplane of C, and T the intersection of H and C. Every extreme point of T is an extreme point of C.

Proof Suppose there exists $x \in T$ such that x is not an extreme point of C, then it is enough to show that it is also not an extreme point of T. If so, there must exist $x_1, x_2 \in C$, $x = \alpha x_1 + (1-\alpha) x_2$. And x must belong to H (intersection), $a^T x = b = \alpha a^T x_1 + (1-\alpha) a^T x_2$. Since C is in one of H's half spaces, suppose C is in H^+, then we have $a^T x_1 \geq b, a^T x_2 \geq b$, $a^T x = b = \alpha a^T x_1 + (1-\alpha) a^T x_2 \geq \alpha b + (1-\alpha) b$. And it must be $a^T x_1 = a^T x_2 = b$. Thus,

x is also not an extreme point of T. ■

Theorem 3.3 (Farkas Lemma (ali_ahmadi_orf_2016))

Let $A \in R^{m\times n}$ and $b \in R^m$. Then exactly one of the following sets must be empty:

1. $\{x \mid Ax = b, x \geq 0\}$
2. $\{y \mid A^T y \leq 0, b^T y > 0\}$

Proof (**ali_ahmadi_orf_2016**)

(ii) to (i). Suppose there exists $Ax = b, x \geq 0$, then we have $x^T A^T y = b^T y > 0$, this contradicts our assumption.

(i) to (ii). Let $a_1, ..., a_n$ denote all columns of A, and $cone\{b_1, ..., b_n\}$ denote the cone of all non-negative combinations. Then C is convex and closed. Let $\{z_k\}$ be a sequence of points in $cone(S)$ converging to a point $\bar{z}$. Considering the following linear program:

$$\min_{\alpha,z} \|z - \bar{z}\|_\infty$$

$$\text{s.t.} \sum_{i=1}^{n} \alpha_i s_i = z$$

$$\alpha_i \geq 0$$

The objective value must be non-negative (norm), for each z_k, there exists α_k that makes the pair (z_k, α_k) feasible to the LP. As z_k get arbitrariliy close to $\bar{z}$, we conclude that the optimal value of this LP is zero. Since LP achieve their optimal values, it follows that $\bar{z} \in cone(S)$.

Suppose there exists b which cannot be represented by A, i.e., $b \notin C$. On the basis of Separating Hyperplane Theorem, the point b and the set C can be (even strictly) separated; i.e.,

$$\exists y \in \mathbb{R}^m, y \neq 0, r \in \mathbb{R} \text{ s.t. } y^T z \leq r \forall z \in C \text{ and } y^T b > r$$

Since $0 \in C$, we must have $r \geq 0$. If $r > 0$, we can replace it by $r' = 0$. For example, in the case of $y^T z > 0$, we can increase α to large enough such that $y^T(\alpha z)$ is also large enough. However, $\alpha z \in C$ contradicts Separating Hyperplane Theorem, thus,

$$y^T z \leq 0, \forall z \in C \text{ and } y^T b > 0$$

Since $a_1, ..., a_n \in C$, we see that $A^T y \leq 0$. ■

Note on *These two sets construct strong alternatives (***ali_ahmadi_orf_2016***), i.e., there is only one set is feasible. By contrast, weak alternatives means at least one set are feasible.*

This theorem is useful to prove that LP is infeasible, if (2) holds, then (1) cannot hold.

Note on Geometric interpretation *Let $a_1, ..., a_n$ denote all columns of A, and $cone\{b_1, ..., b_n\}$ denote the cone of all non-negative combinations. Then only one of two cases will hold: b is in the cone, and b is not in the cone. Thus, we can separate b and the cone with a hyperplane (***ali_ahmadi_orf_2016***).*

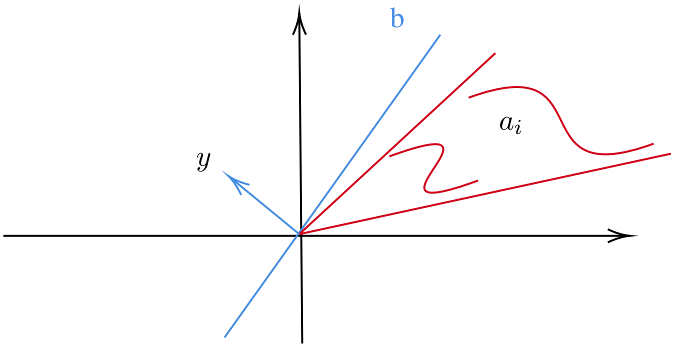

Figure 2: Geometric interpretation of the Farkas lemma

Theorem 3.4 (Farkas Lemma (P. Williamson, 2014, Lec. 7))

Let $A \in R^{m\times n}$ and $b \in R^m$. Then exactly one of the following sets must be empty:

1. $\{x \mid Ax \leq b\}$
2. $\{y \mid A^T y = 0, b^T y < 0, y \geq 0\}$

2' $\{y \mid A^T y = 0, b^T y = -1, y \geq 0\}$

Proof First we prove that (2) iff (2'). The if side is clear. If (2) is true, let $\hat{y} = -\frac{1}{y^T b} y$ and this change (2) to (2').

Secondly, we cannot have both (1) and (2). Suppose otherwise, then we have $b^T y \geq 0$ contradicts our assumption.

Now suppose (1) does not hold, so (2') does not hold either. Define a new system $A^T y = 0, y^T b = -1$ as

$$\bar{A} = \begin{bmatrix} A^T \\ b^T \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$

If (2') holds, there does not exists$z \in R^m$ such that $z \geq 0$ and $\bar{A}z = \bar{b}$. Similarly, on the basis of Separating Hyperplane Theorem, there exists s such that $\bar{A}^T s \geq 0$ and $\bar{b}^T s < 0$. Set s for $x \in R^n$ and $\lambda \in R$.

$$s = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Then $\bar{b}^T s < 0$ implies that

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} < 0$$

which implies that $\lambda > 0$. Also $\bar{A}^T s \geq 0$ implies that

$$\begin{bmatrix} A^T \\ b^T \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0$$

which implies that $Ax + \lambda b \geq 0$ or that $A\left(\frac{-x}{\lambda}\right) \leq b$. Therefore $-x/\lambda$ satisfies (1), so that (1) holds. ■

Definition 3.3 (Polyhedron)

Polyhedron:= $P = \{x \in \mathbb{R}^n : Ax \leq b\}, A \in \mathbb{R}^{m \times n}, m \geq n$

Definition 3.4 (V-polytope, H-polytope (Toth et al., 2017, Ch. 15))

1. *V-polytope: The convex hull of a finite set* $X = \{x^1, \ldots, x^n\}$ *of points in* R^d*,*
$$P = \operatorname{conv}(X) := \left\{ \sum_{i=1}^{n} \lambda_i x^i \mid \lambda_1, \ldots, \lambda_n \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$
2. *H-polytope: The solution set of a finite system of linear inequalities with the extra condition that the set of solutions is bounded.*
$$P = P(A, b) := \left\{ x \in \mathbb{R}^d \mid a_i^T x \leq b_i \text{ for } 1 \leq i \leq m \right\}$$

Note on *Polytope is a bounded polyhedron. Note that the definition in Luenberger and Ye (2015) is different from the main stream, here we adopt the definition from the main stream.*

Definition 3.5 (Bounded Polyhedron)

A polyhedron P *is bounded if* $\exists M > 0$*, such that* $||x|| \leq M$ *for all* $x \in P$*.*

Lemma 3.3 (Main Theorem of Polytope Theory)

The definitions of V-polytopes and H-polytopes are equivalent. That is, every V-polytope has a description by a finite system of inequalities, and every H-polytope can be obtained as the convex hull of a finite set of points (its vertices).

Lemma 3.4 ((P. Williamson, 2014, Lec. 4))

Any polyhedron $P = \{x \in \Re^n : Ax \leq b\}$ *is convex.*

Lemma 3.5 (Minkowski sum of Polytope)

Suppose that $P^i = \left\{x \geq 0 : A^i x = b^i\right\}$ *for* $i = 1, 2$ *are both bounded. Then* $P = P^1 + P^2$ *is also a polytope, where* $P^1 + P^2 = \left\{x^1 + x^2 : x^1 \in P^1 \text{ and } x^2 \in P^2\right\}$*.*

Proposition 3.1 (Open Set and Optimality)

S *is an open set if for each* $x_0 \in S$*, there is an* $\varepsilon > 0$ *such that* $||x - x_0|| < \varepsilon$ *implies that* $x \in S$*. Show that if* S *is an open set, the problem Maximize* $\{c^T x : x \in S\}$*, where* $c \neq 0$*, does not possess an optimal point.*

Proof Suppose for the sake of contradiction that there is an optimal point x_0, we can construct another point $x_0 + \varepsilon c$, where $\varepsilon > 0$, an open feasible region means we can find a small ε to ensure $x_0 + \varepsilon c \in S$, and then show that $x_0 + \varepsilon c$ is optimal than x_0. ■

4 Extreme point, direction and Representation theorem

Definition 4.1 (Extreme Point)

A point x in a convex set C is an extreme point of C if there are no two distinct points x_1, x_2 in C such that $x = \alpha x_1 + (1-\alpha)x_2 \in C$, for some $0 < \alpha < 1$.

Definition 4.2 (Ray)

A collection of points in the form of $\{x_0 + \lambda d : \lambda \geq 0, d \neq 0\}$

Definition 4.3 (Direction of the Set)

A non-zero vector d is a direction of the convex set C if for each $x_0 \in C$, the ray $\{x_0 + \lambda d : \lambda \geq 0, d \neq 0\}$ also belongs to C.

Definition 4.4 (Extreme Direction)

A direction is an extreme direction of C if there are no two distinct directions d_1, d_2 such that $d = \alpha d_1 + (1-\alpha)d_2 \in C$ for some $0 < \alpha < 1$.

Theorem 4.1 (Representation Theorem)

Let $X = \{x : Ax = b, x \geq 0\}$ be a non-empty set. Then the set of extreme points is non-empty and has a finite number of elements, say $x_1, ..., x_k$. The set of extreme directions is empty iff X is bounded. If X is not bounded, then the set of extreme directions is non-empty and has a finite number of elements, say $d_1, ..., d_l$. Moreever, $\overline{x} \in X$ iff it can be represented as a convex combination of $x_1, ..., x_k$ plus a non-negative linear combination of $d_1, .., d_l$, that is,

$$\bar{x} = \sum_{j=1}^{k} \lambda_j x_j + \sum_{j=1}^{l} u_j d_j, \quad \sum_{j=1}^{k} \lambda_j = 1, \lambda_j \geq 0, j = 1, \ldots, k; u_j \geq 0, j = 1, \ldots, l$$

Note on *Representation theorem shows that all solution $\overline{x}$ can be represented in this way. On the basis of this representation, we can derive the optimal solution.*

$$\begin{aligned} \min \sum_{i=1}^{n} c_i x_i = c^T x = c^T\Big(\sum_{j=1}^{k} \lambda_j x_j + \sum_{j=1}^{l} u_j d_j\Big) \\ \iff \min_{\lambda_j, \mu_j} \sum_{j=1}^{k} \lambda_j (c^T x_j) + \sum_{j=1}^{l} u_j (c^T d_j) \end{aligned} \tag{1}$$

s.t.$x \in X$ feasible set

If feasible set is unbounded, $c^T d_j$ can be ≥ 0 or < 0. When $c^T d_j \geq 0$, it is optimal to assign $u_j = 0$. When $c^T d_j < 0$, it is optimal to assign $u_j = -\infty$ (we say the problem is unbounded).

If feasible set is bounded, then there is no such d_j*, i.e., there is no extreme direction. Thus, to optimize the problem, we can find the minimal* $c^T x_j$ *and let* $\lambda_j = 1$ *and* $\lambda_{i \neq j} = 0$*.*

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